Analysis I Lecture 14

Last time:

Started real functions:

Defined . Domain of definition

- Image of function $Im(f) = \{ f(x) \mid x \in D \}$.
- Monotonicity (increasing, decreasing)
- Bounded above / below

Today: - Periodic functions
- odd / even functions
- limit of functions:

Definitions, examples
algebra of limits..

Periodic Functions

Definition $f : D \to \mathbb{R}$.

Then f is called periodic

if $\exists T > 0$ s.t. $\forall x \in D$ s.t.

$x + T \in D$ we have $f(x) = f(x+T)$.

Example Trigonometric functions:

$\alpha > 0$

- $\sin(\alpha x)$, $\cos(\alpha x)$

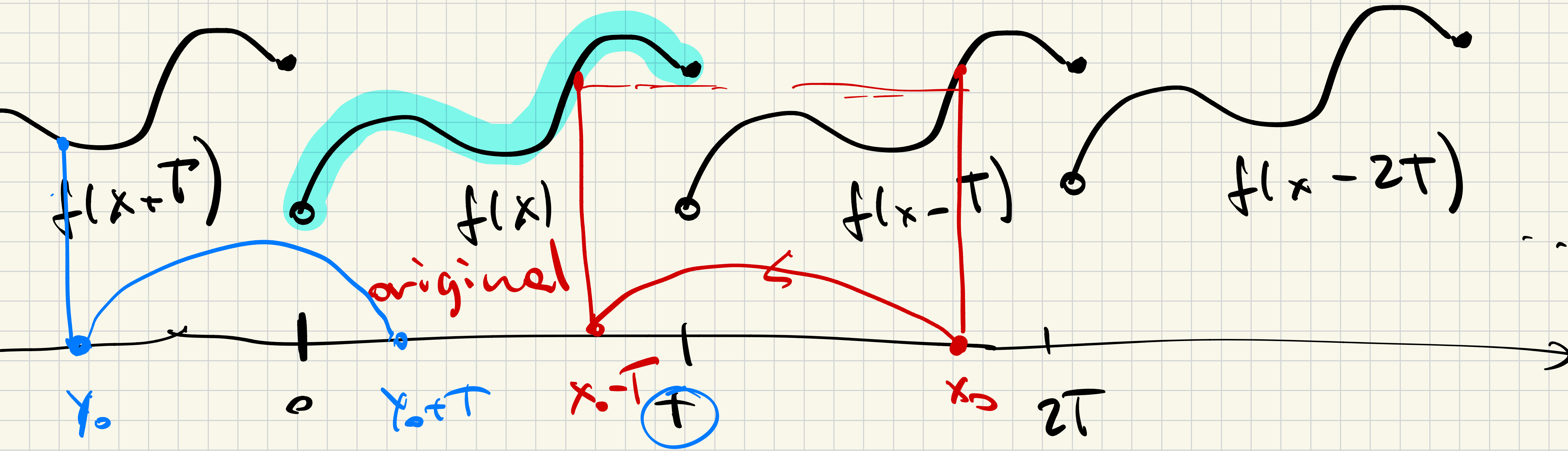
$\frac{2\pi}{\alpha}$ -periodic

e.g. $\sin\left(\alpha\left(x + \frac{2\pi}{\alpha}\right)\right) = \sin(\alpha x + 2\pi)$
$= \sin(\alpha x)$

- $\tan(\alpha x)$, $\cot(\alpha x)$

$\frac{\pi}{\alpha}$ -periodic since $\tan$, $\cot$ are π-periodic.

Remark One can define periodic functions by defining it on some interval and extending by periodicity.



Note that if $f(x+T) = f(x)$ $\forall x$ then

$$f(x+2T) = f((x+T)+T) = f(x+T) = f(x)$$

If T is a period of f then $k \cdot T$ is a period for any $k \in \mathbb{N}$.

Example Define $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ by

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Any rational number $T \in \mathbb{Q}$ is a period of $\chi_{\mathbb{Q}}$:

- if $x \in \mathbb{Q}$ then $x+T \in \mathbb{Q}$

  and hence $\chi_{\mathbb{Q}}(x+T) = \chi_{\mathbb{Q}}(x) = 1$

- if $x \notin \mathbb{Q}$ then $x+T \notin \mathbb{Q}$

  and hence $\chi_{\mathbb{Q}}(x+T) = \chi_{\mathbb{Q}}(x) = 0$

$\chi_{\mathbb{Q}}$ does not have smallest period.

Proposition Let $f: D \to \mathbb{R}$ be T_f-periodic and $g: D \to \mathbb{R}$ be T_g-periodic. Then

1) If $\frac{T_f}{T_g} \in \mathbb{Q}$, $f+g$ and $f \cdot g$ are periodic with domain of definition D.

2) If $\frac{T_f}{T_g} \in \mathbb{Q}$, then f/g is periodic with domain $\underline{\{x \in D \mid g(x) \neq 0\}}$

Proof: Let $\frac{T_f}{T_g} = \frac{a}{b}$ for some $a, b \in \mathbb{N}$

$$\underbrace{T_f \cdot b}_{\text{period of } f} = \underbrace{T_g \cdot a}_{\text{period of } g}$$

$\Rightarrow$

$$(f+g)(x+T_f\cdot a) = f(x+\underline{T_f\cdot b}) + g(x+\underbrace{T_f\cdot b}_{=T_g\cdot a}) =$$

$$= f(x) + g(x) = (f+g)(x).$$

$\blacksquare$

Proposition Let f be periodic, then for any $h(x)$ defined on $\mathrm{Im}(f)$

$h \circ f$ is periodic.

composition of functions defined by

$$h \circ f\,(x) := h\big(f(x)\big)$$

by Definition

Proof: Let T be a period of f

Then T is also a period of $h \circ f$:

$$h\circ f(x+T) = h\big(f(x+T)\big) = h\big(f(x)\big) = h\circ f(x)$$

$\forall x \in D$. $\blacksquare$

Odd / even functions

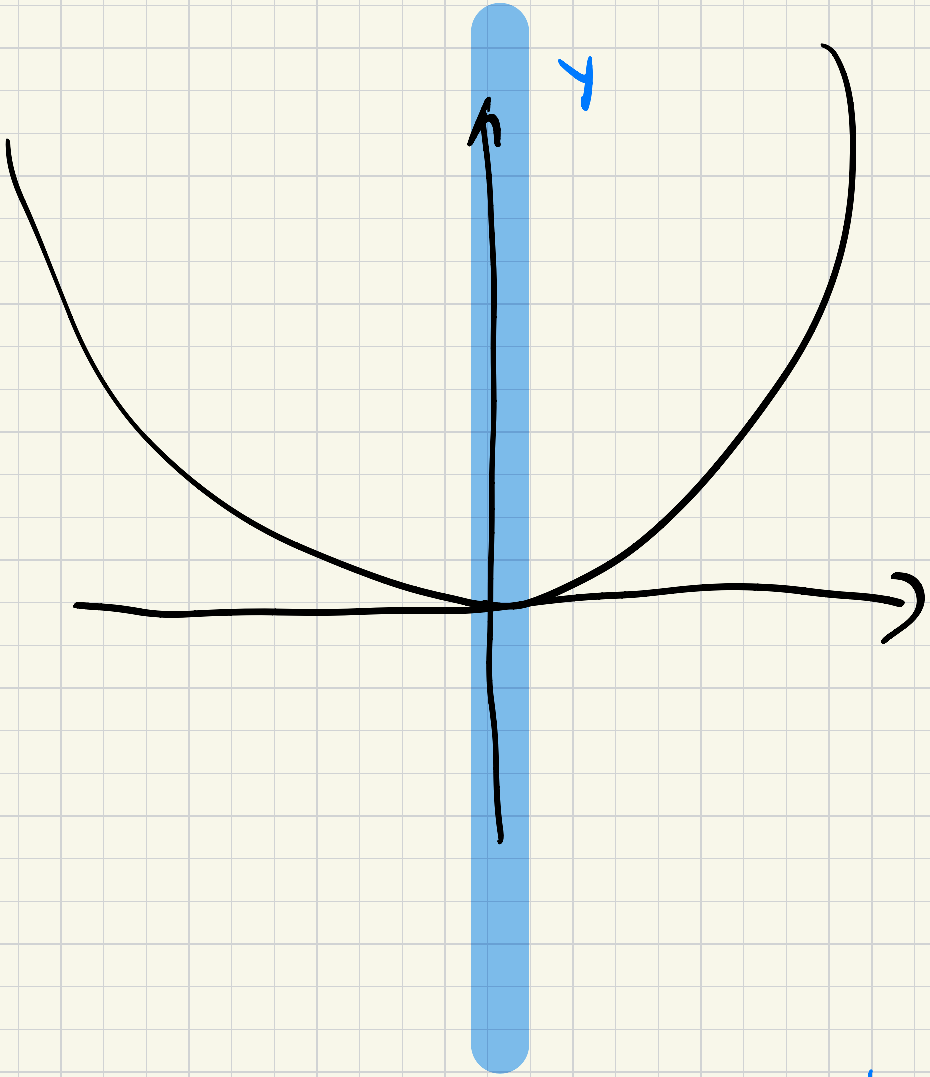
i.e if $x \in D$ then $-x \in D$

Definition Let D be symmetric.

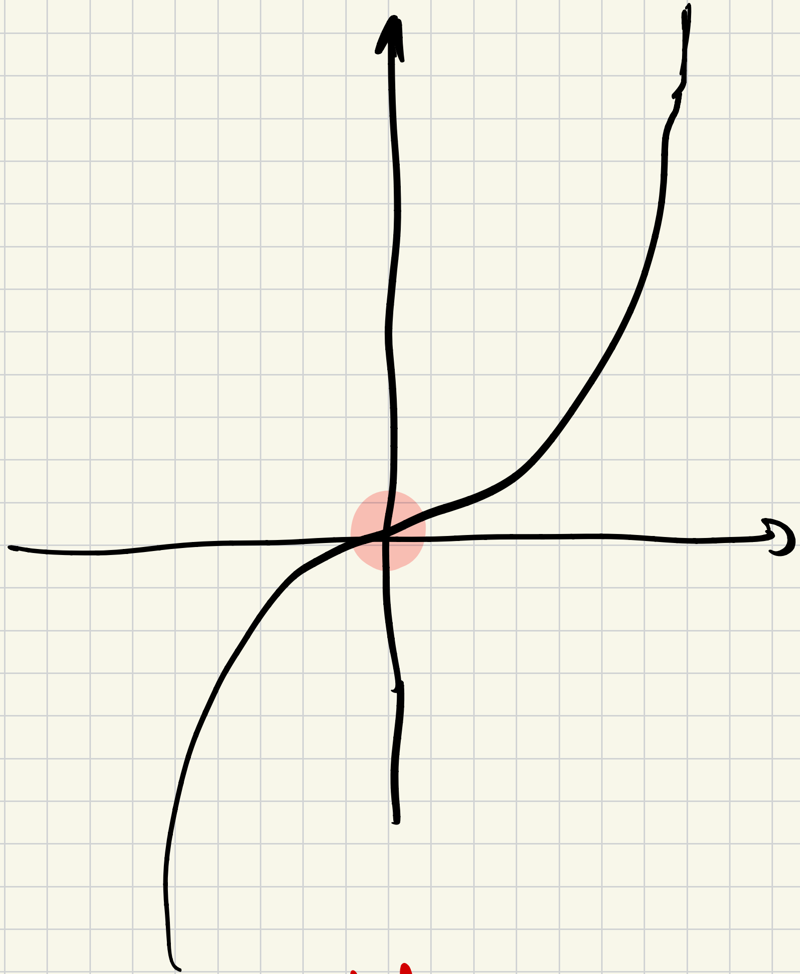
Then $f : D \longrightarrow \mathbb{R}$ is called

1) even if $f(x) = f(-x) \quad \forall x \in D$

2) odd if $f(x) = -f(-x) \quad \forall x \in D$



even symmetric around y-axis



odd symmetric around O.

Examples:

- $\sin(x)$ odd
- $\cos(x)$ even
- $\tan(x)$ odd
- e^x neither
- $\frac{1}{x^2}$ even
- x^3 odd
- $\log(x)$ neither
- $x^2 + x$ Neither

$\hookrightarrow$ $f(1) = 1^2 + 1 = 2$

$f(-1) = (-1)^2 + (-1) = 0$

- $\chi_{\mathbb{Q}}$ even $x \in \mathbb{Q}$

$\Leftrightarrow -x \in \mathbb{Q}$

Fact: Let $p(x) = a_0 + a_1 x + \dots + a_d x^d$ be a polynomial. Then

- $p(x)$ is even iff it has only even degree monomials i.e. $a_{2n+1} = 0$
- $p(x)$ is odd iff it has only odd degree monomials. i.e. $a_{2n} = 0$

Exercise prove this fact.

Example

- $x - x^3 + 5x^5$ is odd
- $1 + x^2 - x^4$ is even
- $x + x^2 + x^3$ is Neither

<u>Remark</u> Given any function $f(x)$ defined on symmetric $D \subset \mathbb{R}$ we can produce

$$p(x) = \frac{f(x) + f(-x)}{2}$$ even

$$q(x) = \frac{f(x) - f(-x)}{2}$$ odd

In particular we get $f(x) = p(x) + q(x)$

<u>Example</u> for $f(x) = e^x$ we get

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$ even

hyperbolic cos

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$ odd.

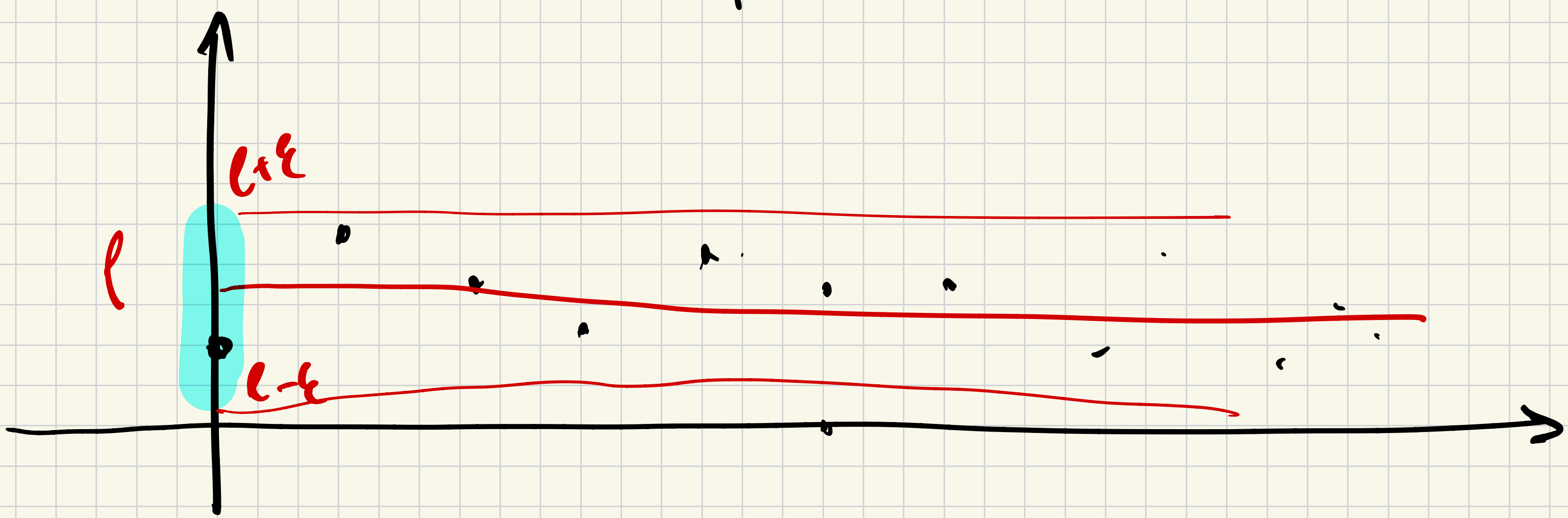
hyperbolic sin

$$\boxed{\text{In particular } \sinh(x) = \cosh(x) = e^x}$$

Limits of functions

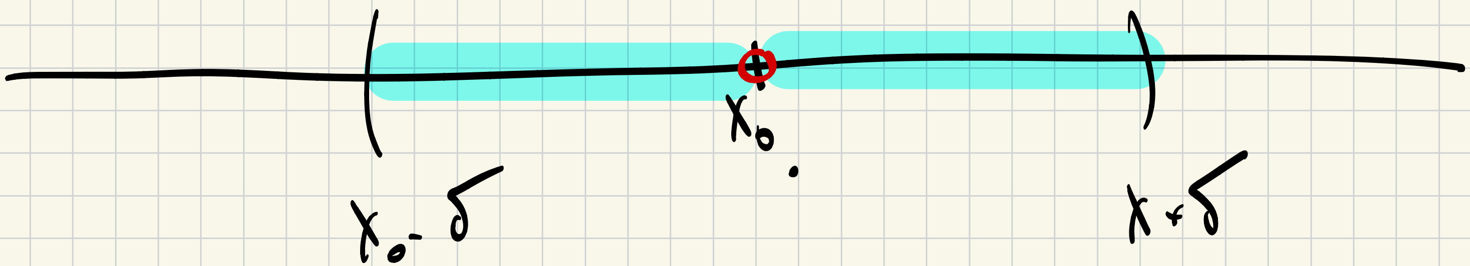
Recall For a sequence (x_n)

$$\lim_{n\to\infty} x_n = \ell \iff \forall \varepsilon > 0 \ \exists N \text{ s.t. } \forall n > N$$
$$|x_n - \ell| < \varepsilon.$$



Now for functions

Define a punctured neighborhood of $x_0 \in \mathbb{R}$ as a set of the form $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ for some $\delta > 0$.



Definition (Limit of a function)

Let $f: D \longrightarrow \mathbb{R}$ be a function and assume that D contains a punctured neighborhood of x_0. Then $\lim_{x \to x_0} f(x) = \ell$

if one of the following two equivalent conditions hold:

Definition ($\lim_{x \to x_0} f(x) = \ell$, continuation)

1) $\forall \varepsilon > 0$ $\exists \delta_\varepsilon > 0$ s.t.

$\forall x \in D$, with $0 < |x - x_0| < \delta_\varepsilon$ we have $|f(x) - \ell| < \varepsilon$.

$x \neq x_0$

2) For every sequence (x_n) with $x_n \subset D \setminus \{x_0\}$

s.t.

$\lim_{n \to \infty} x_n = x_0$, we have $\lim_{n \to \infty} (f(x_n)) = \ell$.

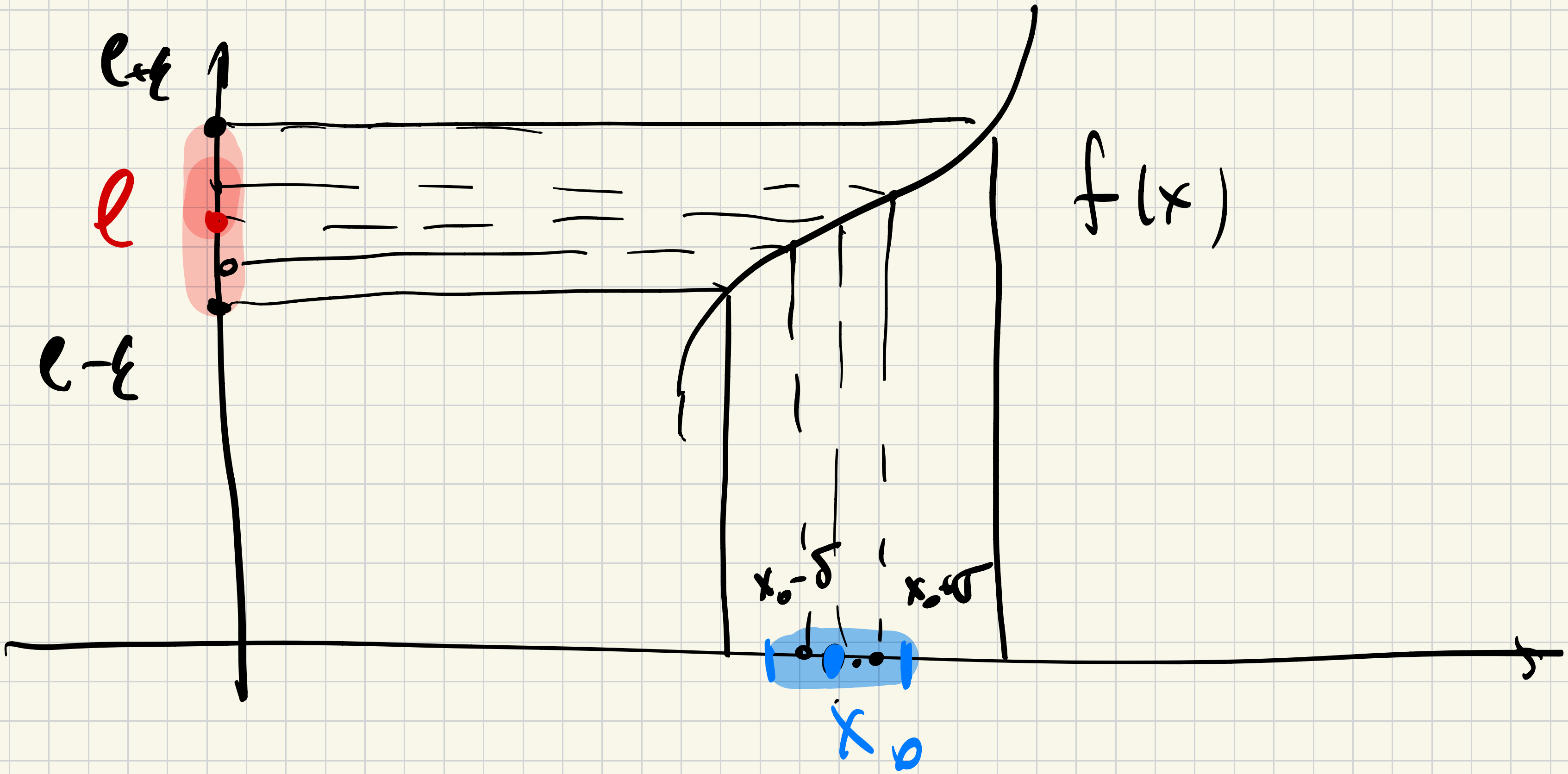
Both definitions informally mean that

fore every value of x close to x_0

the value of $f(x)$ is close to ℓ.

1) $\forall \varepsilon > 0$ $\exists \delta_\varepsilon > 0$ s.t.

$\forall x \in D$, with $0 < |x - x_0| < \delta_\varepsilon$ we have $|f(x) - \ell| < \varepsilon$



Remark: $f(x)$ does not have to be defined at x_0 to have a limit.

E.g. $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$

Defined for every $x \neq 0$

But limit at 0 still exists.

Example of computing a limit:

We show that $\lim_{x \to 2} x^2 = 4$.

- Fix $\varepsilon > 0$. Need to find $\delta > 0$ such that:

if $0 < |x-2| < \delta$ then $|x^2-4| < \varepsilon$

1st definition of the limit

1st way to find δ:

$|x^2-4|<\varepsilon$ ⇐ what we need.

$\Updownarrow$

$-\varepsilon < x^2-4 < \varepsilon$

$x^2-4<\varepsilon \iff x^2 < \varepsilon+4$

$\iff -\sqrt{4+\varepsilon} < x < \sqrt{4+\varepsilon}$

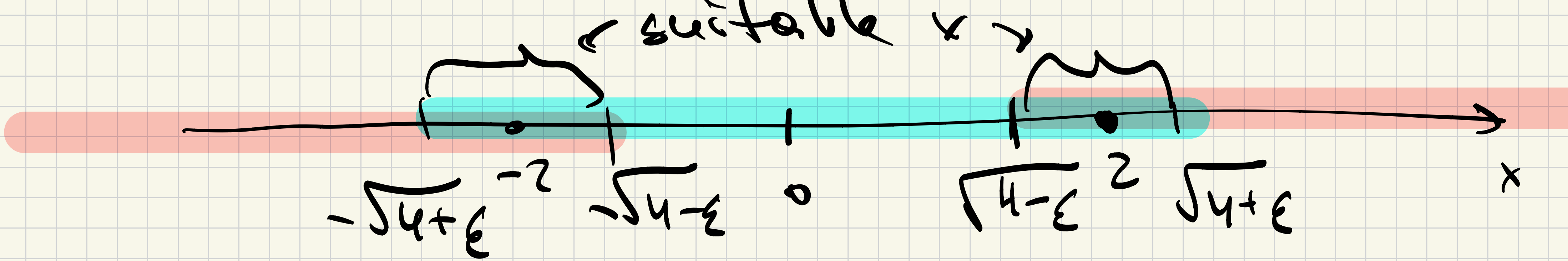
$-\varepsilon < x^2-4 \iff 0 < x^2-(4-\varepsilon)$

$\iff x \in (-\infty, -\sqrt{4-\varepsilon}) \cup (\sqrt{4-\varepsilon}, +\infty)$

Combining two inequalities we get

$$\underline{|x^2-4| < \varepsilon} \iff$$

$$\iff x \in \left(-\sqrt{4+\varepsilon}, \sqrt{4+\varepsilon}\right) \cap \left(\left(-\infty, -\sqrt{4-\varepsilon}\right) \cup \left(\sqrt{4-\varepsilon}, +\infty\right)\right)$$



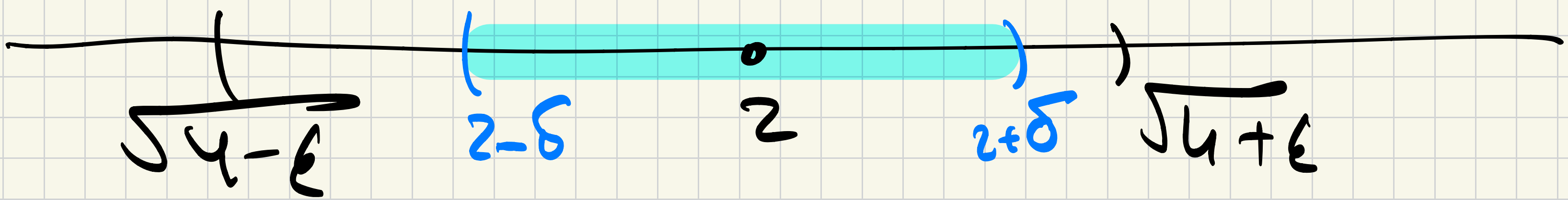
In particular we showed that

for any $x \in (\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon})$

$|x^2-4| < \varepsilon$.

So we can take any

$\delta \leq \min(\sqrt{4+\varepsilon} - 2, 2-\sqrt{4-\varepsilon})$



Remark We don't always have $f(x_0) = \lim_{x \to x_0} f(x)$

e.g $f(x) = \begin{cases} 3 & \text{if } x \neq 0 \\ 7 & \text{if } x = 0 \end{cases}$

then $\lim_{x \to 0} f(x) = 3 \neq 7 = f(0)$.

<u>Proposition</u> (Uniqueness of the limit).

Let $D \subset \mathbb{R}$ contain a punctured neighborhood of x_0 then the limit of $f : D \longrightarrow \mathbb{R}$ at x_0 is unique if exists.

Example How to prove that

limit $\lim_{x \to x_0} f(x)$ does not exist.

Easiest way is to use second definition: We want to find two sequences x_n and x_n' such that

$$\lim_{n\to\infty} x_n = x_0 \qquad \lim_{n\to\infty} x_n' = x_0$$

But $$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} \left(f(x_n')\right)$$

$\Rightarrow$ By uniqueness of $\lim_{x\to x_0} f(x)$ this limit does not exist for

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$x_n = \frac{1}{n}$ notice that

$x_n' = \frac{\sqrt{2}}{n}$ $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n' = 0$

but $\lim_{n\to\infty} \chi_{\mathbb{Q}}\left(\frac{1}{n}\right) = 1 \neq 0 = \lim_{n\to\infty} \chi_{\mathbb{Q}}\left(\frac{\sqrt{2}}{n}\right)$,

So $\lim_{x\to 0} \chi_{\mathbb{Q}}(x)$ does not exist.